Properties of the Dot Product

Let $\vec{a}, \vec{b}, \vec{c}$ be vectors and c a scalar.

1) $\vec{a} \cdot \vec{a} = \|\vec{a}\|^2$ 2) $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$ 3) $\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$

4) $(c\vec{a}) \cdot \vec{b} = c(\vec{a} \cdot b) = \vec{a} \cdot (c\vec{b})$ 5) $\vec{0} \cdot \vec{a} = 0$

Lecture 2

Suppose the angle between two vectors $\vec{u}$ & $\vec{v}$ is θ, then another interpretation of the dot product is:

$$\boxed{\vec{u} \cdot \vec{v} = \|\vec{u}\| \|\vec{v}\| \cos \theta}$$

This can be reversed to find the angle between two vectors $\vec{u}$ & $\vec{v}$:

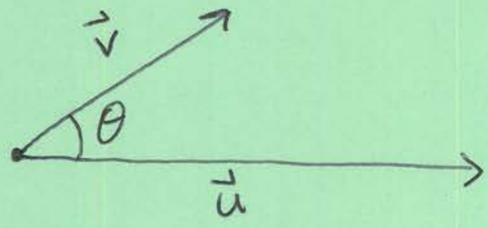
$$\boxed{\theta = \arccos\left(\frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \|\vec{v}\|}\right)}$$

Two vectors are called <u>perpendicular</u> or <u>orthogonal</u> if their dot product is 0 (i.e., $\theta = 90°$)

$$\vec{u} \perp \vec{v} \iff \vec{u} \cdot \vec{v} = 0$$

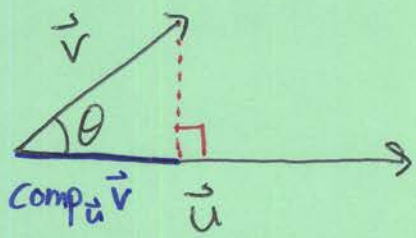
Projections

Let's say we have two vectors $\vec{u}$ & $\vec{v}$ as such



A question we could ask is "how much does $\vec{v}$ point in the direction of $\vec{u}$?" or "what is the piece of $\vec{v}$ in the $\vec{u}$-direction?"

The answer to the first question is the signed length of the blue line, and is called the <u>scalar projection of $\vec{v}$ onto $\vec{u}$</u>: $\text{comp}_{\vec{u}}\vec{v}$



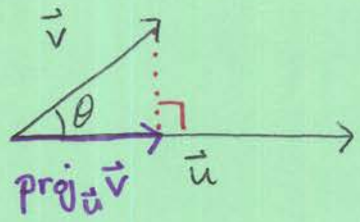
Trigonometry tells us $\text{comp}_{\vec{u}}\vec{v} = \|\vec{v}\| \cos\theta$. Recall that $\vec{u}\cdot\vec{v} = \|\vec{u}\|\|\vec{v}\|\cos\theta$, so:

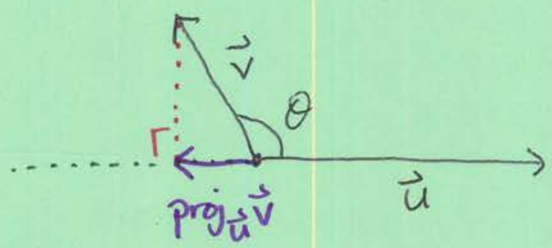
$$\boxed{\text{comp}_{\vec{u}}\vec{v} = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|}}$$

(notice this number is negative if $\theta > 90°$)

The answer to the second question is the vector which is the "shadow" of $\vec{v}$ on $\vec{u}$:

e.g.:





It is called the <u>vector projection of $\vec{v}$ onto $\vec{u}$</u>.

This vector is parallel to $\vec{u}$, and its length is $comp_{\vec{u}}\vec{v}$, so a formula for it is

$$proj_{\vec{u}}\vec{v} = (comp_{\vec{u}}\vec{v})\frac{\vec{u}}{\|\vec{u}\|} = \left(\frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|}\right)\frac{\vec{u}}{\|\vec{u}\|}$$

so

$$\boxed{proj_{\vec{u}}\vec{v} = \left(\frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|^2}\right)\vec{u}}$$

<u>Ex</u>: Find the vector projection of $\vec{v} = \langle 0, 1, \frac{1}{2}\rangle$ onto $\vec{u} = \langle 2, -1, 4\rangle$.

<u>Sol</u>:

$$proj_{\vec{u}}\vec{v} = \left(\frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|^2}\right)\vec{u} = \left(\frac{(2)(0)+(-1)(1)+(4)(\frac{1}{2})}{\left(\sqrt{(2)^2+(-1)^2+(4)^2}\right)^2}\right)\langle 2, -1, 4\rangle$$

$$= \left(\frac{0-1+2}{4+1+16}\right)\langle 2,-1,4\rangle = \frac{1}{21}\langle 2,-1,4\rangle$$

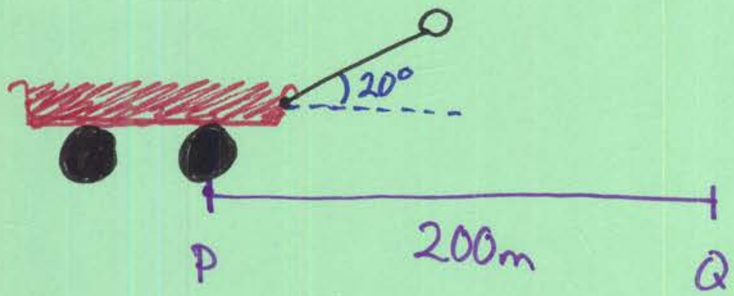
$\diamond$

An Application: Work

Let's say a constant force $\vec{F}$ moves an object from the point P to the point Q. The displacement vector of the object is $\vec{d} = \overrightarrow{PQ}$. The amount of work $\vec{F}$ does in moving the object is the product of the component of $\vec{F}$ in the direction of $\vec{d}$ (i.e., $\text{comp}_{\vec{d}}\vec{F}$) and the displacement distance (i.e., $\|\vec{d}\|$). So, if θ is the angle between $\vec{F}$ & $\vec{d}$, we have:

$$\boxed{\text{Work} = (\text{comp}_{\vec{d}}\vec{F})\|\vec{d}\| = (\|\vec{F}\|\cos\theta)\|\vec{d}\| = \vec{F}\cdot\vec{d}}$$

Example: A child pulls a red wagon a distance of 200m by exerting a force of 100 N at 20° above the horizontal.



How much work has the child done in moving the wagon?

Sol: The amount of work done is:

$$W = (\|\vec{F}\| \cos\theta)\|\vec{d}\| = ((100 \cos 20°)\,N)(200\,m)$$

$$= 20000 \cos 20°\ J = (20 \cos 20°)\,kJ$$

$$\approx 18.794\ kJ$$

◇

12.4 - Cross Product

Suppose we are given two vectors

$$\vec{u} = \langle u_1, u_2, u_3 \rangle \ \& \ \vec{v} = \langle v_1, v_2, v_3 \rangle .$$

We would like to create a new vector, $\vec{w} = \langle w_1, w_2, w_3 \rangle$, out of them so that $\vec{u}, \vec{v} \perp \vec{w}$. The desired conditions give us two equations:

$$\begin{cases} \vec{u} \cdot \vec{w} = 0 \\ \vec{v} \cdot \vec{w} = 0 \end{cases}$$

This actually has a whole family of solutions, one of which is

$$\vec{w} = \langle u_2 v_3 - v_3 u_2 ,\ u_3 v_1 - u_1 v_3 ,\ u_1 v_2 - u_2 v_1 \rangle .$$

$\vec{w}$ is called the <u>cross product</u> of $\vec{u}$ & $\vec{v}$ and is written $\vec{u} \times \vec{v}$. We have a simpler way of computing cross products than solving the above system or memorizing the above formula. It uses <u>determinants</u>. The cross product is also called the <u>vector product</u>.

Determinants

- of a 2×2 matrix

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

- of a 3×3 matrix

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

$$= a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1)$$

Using the unit vector notation, we can write the cross product as

$$\boxed{\vec{u}\times\vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \hat{i}\begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} - \hat{j}\begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} + \hat{k}\begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}}$$

⚠ Whereas the dot product can be taken any two vectors of the same dimension, the cross product ONLY makes sense in dimension 3!

Ex: Find the cross product of

$$\vec{u} = \langle 1, 3, -2 \rangle \ \& \ \vec{v} = \langle 2, 4, 6 \rangle$$

Sol:

$$\vec{u} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 3 & -2 \\ 2 & 4 & 6 \end{vmatrix} = \hat{i}(18-(-8)) - \hat{j}(6-(-4)) + \hat{k}(4-6)$$

$$= \langle 26, -10, -2 \rangle$$

◇

Before, to check whether two nonzero vectors are parallel, we needed to find a constant c such that $\vec{u} = c\vec{v}$. The cross product gives us an easier way:

Theorem: Two nonzero vectors $\vec{u}$ and $\vec{v}$ are parallel if and only if $\vec{u} \times \vec{v} = \vec{0}$

Proof: If $\vec{u} \parallel \vec{v}$, then $\vec{u} = c\vec{v}$ for some $c \in \mathbb{R}$.
So $\vec{u} = c\vec{v} = \langle cv_1, cv_2, cv_3 \rangle$, thus

$$\vec{u} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ cv_1 & cv_2 & cv_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = (cv_2v_3 - cv_3v_2)\hat{i} - (cv_1v_3 - cv_3v_1)\hat{j} + (cv_1v_2 - cv_2v_1)\hat{k} = \vec{0}$$

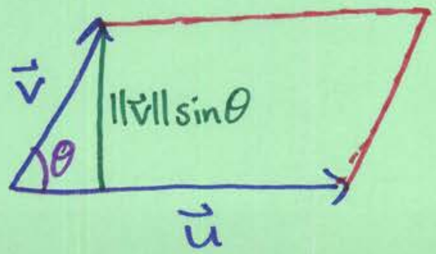
$\square$

By simply writing out $\|\vec{u} \times \vec{v}\|$, we also get:

Theorem: If θ is the angle between $\vec{u}$ & $\vec{v}$ (so $0 \le \theta \le \pi$), then

$$\boxed{\|\vec{u} \times \vec{v}\| = \|\vec{u}\| \, \|\vec{v}\| \sin\theta}$$

This theorem actually also has a nice geometrical application:
Given two vectors $\vec{u}$ & $\vec{v}$, we get the parallelogram that they span:

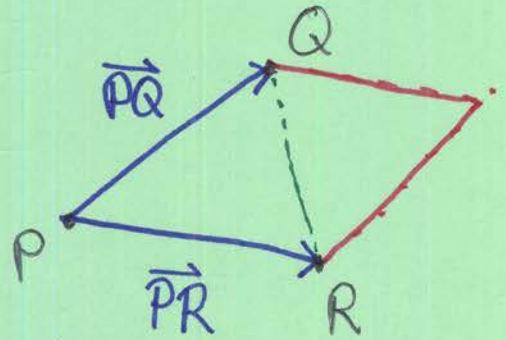


the area of which is $A = \|\vec{u}\| \, \|\vec{v}\| \sin\theta$

The area of the parallelogram spanned by two vectors $\vec{u}$ & $\vec{v}$ is $\|\vec{u}\times\vec{v}\|$.

Ex: Find the area of the triangle with vertices $P=(0,0,-3)$, $Q=(4,2,0)$, and $R=(3,3,1)$

Sol: Say the points are arranged as



Notice that the triangle ΔPQR has half the area of the parallelogram spanned by $\vec{PQ}$ & $\vec{PR}$. So,

$$\text{Area of } \Delta PQR = \frac{1}{2}\|\vec{PQ}\times\vec{PR}\|$$

$\vec{PQ}=Q-P=\langle 4,2,3\rangle$, $\vec{PR}=\langle 3,3,4\rangle$

$$\text{Area}=\frac{1}{2}\left\| \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & 2 & 3 \\ 3 & 3 & 4 \end{vmatrix} \right\| = \frac{1}{2}\|(8-9)\hat{i}-(16-9)\hat{j}+(12-6)\hat{k}\| = \frac{1}{2}\|\langle -1,-7,6\rangle\|$$

$$=\frac{1}{2}\sqrt{1+49+36}=\frac{1}{2}\sqrt{86} \qquad \diamond$$

Using the properties of the cross product we have so far, we have the following:

$$\hat{i} \times \hat{j} = \hat{k}, \quad \hat{j} \times \hat{k} = \hat{i}, \quad \hat{k} \times \hat{i} = \hat{j}$$

$$\hat{j} \times \hat{i} = -\hat{k}, \quad \hat{k} \times \hat{j} = -\hat{i}, \quad \hat{i} \times \hat{k} = -\hat{j}$$

This can be remembered as a cyclic property:

$$\begin{tikzcd}
& \hat{i} \arrow[dr, bend left] & \\
\hat{k} \arrow[ur, bend left] & & \hat{j} \arrow[ll, bend left]
\end{tikzcd}$$

Moving in the direction of the arrows, no problem; moving against the arrows creates a minus sign in the answer.

Notice that this establishes that $\times$ is <u>not</u> commutative.

Observe this:

$$(\hat{i} \times \hat{i}) \times \hat{j} = \vec{0} \times \hat{j} = \vec{0} \quad \& \quad \hat{i} \times (\hat{i} \times \hat{j}) = \hat{i} \times \hat{k} = -\hat{j}$$

showing that $\times$ isn't even associative!

So, what properties are true?

Properties: Let $\vec{a}, \vec{b}$, and $\vec{c}$ be vectors & c a scalar.

1) $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$, 2) $(c\vec{a}) \times \vec{b} = c(\vec{a} \times \vec{b}) = \vec{a} \times (c\vec{b})$

3) $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$, 4) $(\vec{a} + \vec{b}) \times \vec{c} = \vec{a} \times \vec{c} + \vec{b} \times \vec{c}$

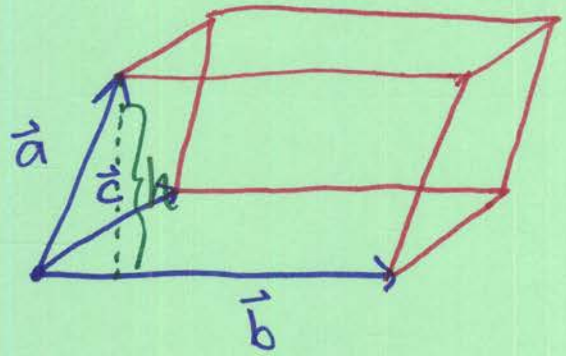
5) $\vec{a} \cdot (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \cdot \vec{c}$, 6) $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$

Triple Product: Given 3 vectors $\vec{u}, \vec{v}$, and $\vec{w}$, the triple scalar product, is the product $\vec{u} \cdot (\vec{v} \times \vec{w})$, a scalar, and can be computed as a determinant with the 3 vectors as rows:

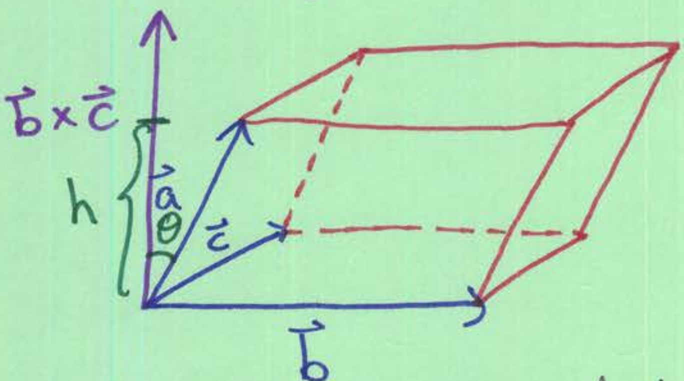
$$\boxed{\vec{u} \cdot (\vec{v} \times \vec{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}}$$

A valid question to ask is "what is the purpose of this product?" The point is the following:

Just as 2 non-parallel, non-zero vectors span a parallelogram, 3 such vectors (in this they need to be pairwise non-parallel, which means non-coplanar) will span a parallelepiped:



The volume of a parallelepiped is $Vol = A \cdot h$ where A is the area of the base and h is the height. We already know $A = \|\vec{b} \times \vec{c}\|$. We can find h with a little geometry:



$h = \|\vec{a}\| \, |\cos\theta|$ (we need to use $|\cos\theta|$ in case $\theta > \frac{\pi}{2}$, e.g., if $\vec{b}$ and $\vec{c}$ were switched in the picture)

So, $\text{Vol} = A \cdot h = \|\vec{b} \times \vec{c}\| \left(\|\vec{a}\| \, |\cos\theta|\right)$, which gives us

<u>Volume of a parallelepiped</u>: The volume of the parallelepiped spanned by the vectors $\vec{a}, \vec{b}$, and $\vec{c}$ is

$$\boxed{\text{Vol} = |\vec{a} \cdot (\vec{b} \times \vec{c})|}$$

This triple product has another use: checking whether 3 vectors are coplanar. Think about it geometrically; if the 3 vectors are coplanar, then the volume of the parallelepiped should be 0 since there is no "third direction". This gives us:

Three nonzero vectors $\vec{u}, \vec{v}$, and $\vec{w}$ are <u>coplanar</u> if and only if $\vec{u} \cdot (\vec{v} \times \vec{w}) = 0$.

<u>Ex</u>: Determine whether $\vec{u} = \langle 1, 5, -2 \rangle$, $\vec{v} = \langle 3, -1, 0 \rangle$, and $\vec{w} = \langle 5, 9, -4 \rangle$ are coplanar.

<u>Sol</u>

$$\vec{u} \cdot (\vec{v} \times \vec{w}) = \begin{vmatrix} 1 & 5 & -2 \\ 3 & -1 & 0 \\ 5 & 9 & -4 \end{vmatrix} = 1(4-0) - 5(-12-0) - 2(27+5) = 4 + 60 - 64 = 0$$

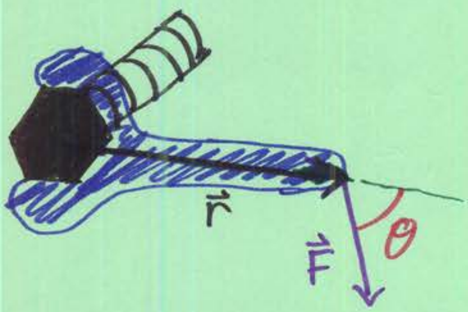
$\Rightarrow$ The vectors are coplanar. ◇

An application Torque

Torque is created by applying a force to an object at a point given by a position vector, for example using a wrench to tighten a bolt.

Torque is a measure of the tendency of the object to rotate about a pivot point (from which the position vector radiates). If the position vector is $\vec{r}$ and the force is $\vec{F}$, the torque vector is

$$\boxed{\vec{\tau} = \vec{r} \times \vec{F}}$$



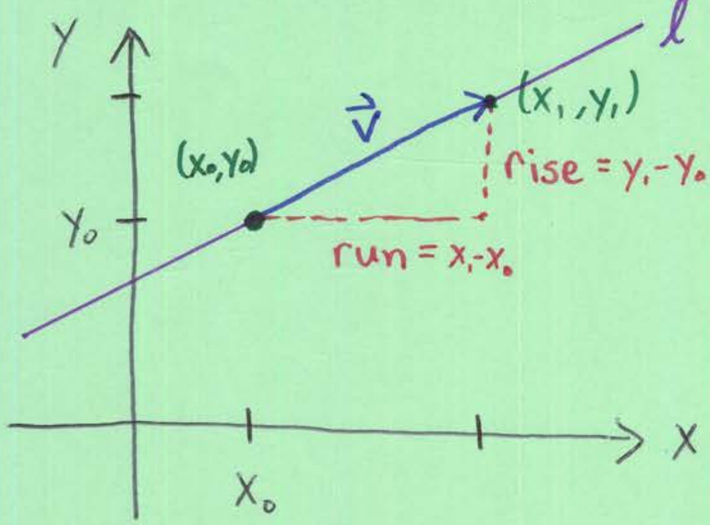
$$\|\vec{\tau}\| = \|\vec{r}\|\,\|\vec{F}\| \sin\theta$$

12.5 - Lines and Planes

Let's look back at how we describe a line in the plane: we use the slope (read: direction) of the line and a point on the line:

$$y - y_0 = m(x - x_0)$$

The slope, $m = \frac{\text{rise}}{\text{run}}$, and notice we can encode that as a vector: $\vec{v} = \langle \text{run}, \text{rise} \rangle$ as follows:



We can see then that any multiple of $\vec{v}$ starting at (x_0, y_0) points to a point on ℓ. This gives us the vector equation for the line:

$$\vec{\ell} = \vec{P}_0 + t\vec{v}, \quad \vec{P}_0 = \langle x_0, y_0 \rangle, \quad \vec{\ell} = \langle x, y \rangle$$

If we use $\vec{v} = \langle 1, m \rangle$, then

$$\langle x, y \rangle = \vec{l} = \vec{P}_0 + t\vec{v} = \langle x_0, y_0 \rangle + \langle t, mt \rangle$$

$$\begin{cases} x = x_0 + t \\ y = y_0 + mt \end{cases} \overset{(m \neq 0)}{\Longrightarrow} \begin{cases} t = x - x_0 \\ t = \frac{1}{m}(y - y_0) \end{cases} \Rightarrow \frac{1}{m}(y - y_0) = x - x_0$$

$$\Rightarrow y - y_0 = m(x - x_0)$$

a familiar face

In 3 dimensions, the equation for a line looks exactly the same:

direction vector: $\vec{v} = \langle a, b, c \rangle$

position vector of point on line: $\vec{P}_0 = \langle x_0, y_0, z_0 \rangle$

$$\boxed{\text{Vector equation of line}: \vec{l} = \vec{P}_0 + t\vec{v}}$$

If we write this out:

$$\langle x, y, z \rangle = \vec{l} = \vec{P}_0 + t\vec{v} = \langle x_0, y_0, z_0 \rangle + t\langle a, b, c \rangle$$

We can separate it into 3 equations

$$\begin{cases} x = x_0 + at \\ y = y_0 + bt \\ z = z_0 + ct \end{cases}$$

called the <u>parametric equations</u> of the line.

<u>Ex</u>: The the vector and parametric equations for the line passing through $(-2,4,0)$ and $(1,1,1)$.

<u>Sol</u>: First, we need a direction vector for the line. If $P=(-2,4,0)$ & $Q=(1,1,1)$, a direction vector is $\vec{v} = \overrightarrow{PQ} = \langle 3,-3,1\rangle$. So, a vector equation for the line is: $\vec{l} = \overrightarrow{OP} + t\vec{v} = \langle -2,4,0\rangle + t\langle 3,-3,1\rangle$

From this, we can read off the parametric equations

$$\begin{cases} x = -2+3t \\ y = 4-3t \\ z = t \end{cases}$$

$\diamond$